# Introducing Parallel Prefix-Sums Algorithm and Its Applications in An Undergraduate Course

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***Abstract*—The parallel prefix-sums algorithm and its variations are often used as key substeps in more complex parallel computations. We have successfully introduced the parallel prefix-sum algorithm and its key ideas in a senior level undergraduate Computer Science course. To improve the understanding of these ideas, students were required to solve several problems using simple variations of the basic prefix-sums algorithm. They were also required to solve several non-trivial problems, including their 2-dimensional generalizations. We briefly discuss here four problems of this second category.**

***Index Terms*—prefix-sums, parallel algorithm, applications**

## I. THE PREFIX-SUMS

Let $x[0..(N-1)]$, in short $x[\,]$ or simply $x$ when no confusion is likely, be an array of numbers. We call the sum $s[j] = \sum_{i=0}^{j} x[i]$ the $j^{th}$ (inclusive) prefix-sum of $x$. The prefix-sums of $x = [3, 12, -4, 8, 2]$ then equals $s = [3, 15, 11, 19, 21]$. Many computation problems involve prefix-sums in some form [1]-[4]. For example, if $x[j]$ is the probability of #(heads) = $j$ in $N-1$ tosses of a coin, then $s[j]$ is the probability of #(heads) $\leq j$. An optimal sequential algorithm to compute the prefix-sums is to let $s[0] = x[0]$ and $s[j] = s[j-1] + x[j]$ for $1 \leq j < N$. This takes $O(N)$ time. A parallel algorithm for computing the prefix-sums takes $O(logN)$ time using $O(N)$ CPU's or computing agents [1]-[2].

We define prefix-mins of $x$ to be the array $m$, where $m[j]$ = $\min\{x[i] : 0 \leq i \leq j\}$. The prefix-maxs of $x$ is defined similarly, and each of these can be computed in time $O(logN)$ as well using $O(N)$ agents. In what follows, we use the convention $z[-1] = 0$ for any array $z[0..(N-1)]$.

## II. THE MAXIMUM CONSECUTIVE SUM PROBLEM: MCS

For a given $x$, we want to find a subarray $x[i..j]$ of $x$ with maximum sum of items $s(i,j)$ = $\sum_{k=i}^{j} x[k] = s[j] - s[i-1]$, where $s$ is the prefix-sums of $x$. Let $MCS(x)$ be the "leftmost" subarray $x[i..j]$ having the maximum $s(i,j)$, with smallest possible $j$ and the largest $i \leq j$ for that $j$. Also, let $\sigma MCS(x)$ be the sum of items in $MCS(x)$. Let $m$ be the prefix-mins of $s$ and $s_m[j] = s[j] - m[j-1]$, with $s_m[0] = s[0] = x[0]$. Then, $\sigma MCS(x)$ = $\max\{s(i,j) : i \leq j\}$ = $\max\{s_m[j] : 0 \leq j < N\}$. Thus, $\sigma MCS(x)$ can be computed in $O(logN)$ time. It is not hard to see that $MCS(x)$ can also be computed in $O(logN)$ time.

**Example 1.** Fig. 1 shows an $x$ and its associated $s$, $m$, and $s_m$. Here, $MCS(x) = x[4..7]$ = [11, −2, −6, 12] and $\sigma MCS(x) = s[7] - m[6] = 10 - (-5) = 15$. Fig. 1 also shows the array items $indx[j]$ = $\max\{i : 0 \leq i \leq j$ and $s[i] = m[j]\}$. This is needed to find $i$ in $x[i..j] = MCS(x)$. We compute $indx[j]$'s as we compute $m[j]$'s. Note that $j$ in $x[i..j] = MCS(x)$ equals $\min\{j' : s_m[j'] = \sigma MCS(x)\}$. ■

| | $x[0]$ | $x[1]$ | $x[2]$ | $x[3]$ | $x[4]$ | $x[5]$ | $x[6]$ | $x[7]$ | $x[8]$ | $x[9]$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $x$: | −2 | 1 | 3 | −7 | 11 | −2 | −6 | 12 | −3 | −1 |
| $s$: | −2 | −1 | 2 | −5 | 6 | 4 | −2 | 10 | 7 | 6 |
| $m$: | −2 | −2 | −2 | −5 | −5 | −5 | −5 | −5 | −5 | −5 |
| $s_m$: | −2 | 1 | 4 | −3 | 11 | 9 | 3 | 15 | 12 | 11 |
| $indx$: | 0 | 0 | 0 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |

Fig. 1. Illustration of the arrays $s$, $m$, $s_m$ and $indx$ for a given array $x$.

## III. LARGEST BLOCK OF CONSECUTIVE 1'S: LBO

Assume that each $x[i] = 0$ or 1, at least one $x[i] = 0$, and at least one $x[i] = 1$, i.e., $0 < s[N-1] < N$, where $s$ is the prefix-sums of $x$. We can thus test the above property by computing $s$ in $O(logN)$ time. We want to find a largest length subarray $x[i..j], i \leq j$, of $x$ consisting of only 1's. Let $LBO(x)$ denote the "leftmost" such a subarray $x[i..j]$, with $j$ as small as possible and the largest $i \leq j$ for that $j$. Also, let $\sigma LBO(x)$ denote length($LBO(x)$) = $j - i + 1$ = the sum of items in $LBO(x)$. Thus, $x = [1, 0, 1, 1, 1, 0, 1, 1, 1]$ gives $LBO(x) = x[2..4]$ and $\sigma LBO(x) = 3$. If we write $x'$ for the array obtained by replacing each 0 in $x$ by $-N$, then it is easy to see that $MCS(x') = x'[i..j]$ if and only if $LBO(x) = x[i..j]$ and hence $\sigma MCS(x') = \sigma LBO(x)$. Using $O(N)$ agents, we can determine $x'$ from $x$ in $O(1)$ time and thus we can compute $LBO(x)$ and $\sigma LBO(x)$ in time $O(logN)$ using $O(N)$ agents.

## IV. TWO DIMENSIONAL MCS: MCS2

This is a generalization of the $MCS$-problem in $\S II$ to 2-dimensional arrays. For an $M \times N$ matrix $x$, $0 \leq r_1 \leq r_2 < M$, and $0 \leq c_1 \leq c_2 < N$, let $x(r_1, r_2, c_1, c_2)$ be the submatrix of $x$ consisting of rows $r_1$ to $r_2$ and columns $c_1$ to $c_2$. We want to find an $x(r_1, r_2, c_1, c_2)$ such that its sum of items $s(r_1, r_2, c_1, c_2) = \sum_{i=r_1}^{r_2} \sum_{j=c_1}^{c_2} x[i,j]$ is maximum. Let $MCS2(x)$ denote an optimum submatrix of $x$ and as usual we want to find an $x(r_1, r_2, c_1, c_2) = MCS2(x)$ with the smallest $r_2$, the largest $r_1$ for that $r_2$, the smallest $c_2$ for that

$r_1$ and $r_2$, etc. Let $\sigma MCS2(x)$ be the sum of items of an $MCS2(x)$. Without loss of generality, we assume $M \leq N$.

**Example 2.** For the matrix $x$ in Fig. 2, each subarray of column 3 has its sum of items negative if it contains $-5$. This implies that $MCS2(x)$ does not contain the item $-5$ and that means it equals row 0, i.e., $x(0,0,0,3)$, or row 2, i.e., $x(2,2,0,3)$, or all 3 rows and the first 3 columns of $x$, i.e., $x(0,2,0,2)$. Thus, $MCS2(x) = x(0,2,0,2)$ and $\sigma MCS2(x) = 13$.

We solve the $MCS2$ problem by reducing it to a set of 1-dimensional $MCS$ problems by considering groups of consecutive rows of $x$ and then taking the best solution of those 1-dimensional $MCS$ problems. For $0 \leq r_1 \leq r_2 < M$, let $x_{r_1,r_2}$ be the 1-dimensional array of column-wise sums of items in rows $r_1$ to $r_2$, i.e., $x_{r_1,r_2}[j] = \sum_{i=r_1}^{r_2} x[i,j]$. Clearly, $\sigma MCS(x_{r_1,r_2}) = \sigma MCS2(x)$ if and only if $\sigma MCS(x_{r_1,r_2}) = \max\{\sigma MCS(x_{r_1',r_2'}) : 0 \leq r_1' \leq r_2' < M\}$. To compute $MCS2(x)$, we simply choose smallest $r_2$ and largest $r_1 \leq r_2$ such that $\sigma MCS(x_{r_1,r_2}) = \sigma MCS2(x)$. We can efficiently compute all $x_{r_1,r_2}$ as follows. Let $y_c$ be the column $c$ of $x$, i.e., $y_c[i] = x[i,c]$ and let $s_c$ be the prefix-sums of $y_c$. Also, let $x'$ denote the $M \times N$ matrix whose columns are $y_c' = s_c$ and let $x_r'$ be the row $r$ of $x'$. Thus, $x_r'[j] = \sum_{i=0}^{r} x[i,j] = x_{0,r}[j]$ and hence $x_{r_1,r_2} = x_{r_2}' - x_{r_1-1}'$, i.e., $x_{r_1,r_2}[j] = x_{r_2}'[j] - x_{r_1-1}'[j]$, where $x_{-1}'[j] = 0$ by convention. See Fig. 2.

Using $O(MN)$ agents, we can compute in parallel all prefix-sums $s_c, 0 \leq c < N$, and hence the matrix $x'$ in $O(logM)$ time. For each fixed $r_1$, we can compute all $x_{r_1,r_2}$ in parallel in time $O(1)$ and then compute all the related $MCS(x_{r_1,r_2})$ and $\sigma MCS(x_{r_1,r_2})$ in time $O(logN)$ based on the results in § II. This gives a total time $O(MlogN)$ time as we vary $r_1$. Finally, the number of $(r_1,r_2)$-combinations is $M(M+1)/2 \leq MN$ and hence the best of all $\sigma MCS(x_{r_1,r_2})$ and its associated $MCS(x_{r_1,r_2})$ can be computed in time $O(log(MN))$. This gives the total time $O(MlogN)$ for $MCS2(x)$ and $\sigma MCS2(x)$ using $O(MN)$ agents. ■

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 2 & -5 \\ 1 & 3 & 1 & 0 \end{bmatrix}$$

(i) A matrix $x$.

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & 1 & 4 & -4 \\ 4 & 4 & 5 & -4 \end{bmatrix}$$

(ii) The matrix $x'$ of prefix-sums of columns of $x$.

| $r_1$ | $r_2$ | $x_{r_1,r_2}$ | $MCS(x_{r_1,r_2})$ | $\sigma MCS(x_{r_1,r_2})$ |
|---|---|---|---|---|
| 0 | 0 | [1, 0, 2, 1] | $x_{0,0}[0..3]$ | 4 = 1+0+2+1 |
| | 1 | [3, 1, 4, −4] | $x_{0,1}[0..2]$ | 8 = 3+1+4 |
| | 2 | [4, 4, 5, −4] | $x_{0,2}[0..2]$ | 13 = 4+4+5 |
| 1 | 1 | [2, 1, 2, −5] | $x_{1,1}[0..2]$ | 5 = 2+1+2 |
| | 2 | [3, 4, 3, −5] | $x_{1,2}[0..2]$ | 10 = 3+4+3 |
| 2 | 2 | [1, 3, 1, 0] | $x_{2,2}[0..2]$ | 5 = 1+3+1 |

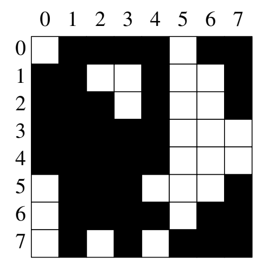
(iii) $\sigma MCS2(x)$ = max of $\sigma MCS(x_{r_1,r_2})$'s = 13 = $\sigma MCS(x_{0,2})$ and $MCS2(x) = x(0,2,0,2)$ because $MCS(x_{0,2}) = x_{0,2}[0..2]$.

Fig. 2. Illustration of computation of $MCS2(x)$ and $\sigma MCS2(x)$.

## V. An image processing application of MCS2

Fig. 3(i) shows an $8 \times 8$ black-and-white image. We want to find a largest size rectangular black area in such an image. The rows 3 to 6 and columns 1 to 3 gives a largest black rectangular area of size 12 in Fig. 3(i).

An $N \times N$ image can be represented by an $N \times N$ matrix $x$, where $x[i,j] = 1$ or 0 according as the pixel (unit square) in row $i$ and column $j$ is black or white. One way to solve this problem is by converting it to a 2-dimensional version of $LBO$ problem in $\S V$. Hence, we can call this problem $LBO2$. Let $x'$ be the matrix obtained by replacing each 0 in $x$ by $-N^2$. Assuming that at least one pixel in $x$ equals 1, a solution of $MCS2(x')$ would not involve a 0-pixel of $x$. Thus, $MCS2(x')$ gives a solution of $LBO2(x)$, i.e., a largest area black rectangle of $x$. This takes $O(N.logN)$ time using $N^2$ agents based on the results in $\S IV$.



(i) An 8×8 black-and-white image $x$; $N$ = 8.

$$\begin{bmatrix} -64 & 1 & 1 & 1 & 1 & -64 & 1 & 1 \\ 1 & 1 & -64 & -64 & 1 & -64 & -64 & 1 \\ 1 & 1 & 1 & -64 & 1 & -64 & -64 & 1 \\ 1 & 1 & 1 & 1 & 1 & -64 & -64 & -64 \\ 1 & 1 & 1 & 1 & 1 & -64 & -64 & -64 \\ -64 & 1 & 1 & 1 & -64 & -64 & -64 & 1 \\ -64 & 1 & 1 & 1 & 1 & -64 & 1 & 1 \\ -64 & 1 & -64 & 1 & -64 & 1 & 1 & 1 \end{bmatrix}$$

(ii) The matrix $x'$; black pixels correspond to 1 and balck pixels to −64 = $-N^2$.

Fig. 3. The matrix $x'$ of a black-and-white image $x$; $MCS2(x')$ equals rows 3 to 6 and columns 1 to 3, giving a largest black rectangular area in $x$.

## VI. Conclusion

We have presented here four problems, involving the applications of the basic prefix-sums computation, that we successfully used for introducing parallel computation in a senior-level undergraduate Computer Science course. The $MCS$-problem is the core of these problems, and the students needed several hints to arrive at its solution. The students could easily solve the $LBO$-problem given the hint to convert it to the $MCS$-problem by modifying the original input array. The students could solve the $MCS2$-problem, which is a 2-dimensional generalization of the $MCS$-problem, by applying $MCS$ to groups of rows of the input matrix. Finally, the image processing problem is a 2-dimensional generalization of the $LBO$ problem and the students could easily solve it.

## References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